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SOME APPLICATIONS OF MATHEMATICS TO EDUCATIONAL STATISTICS.*

BY GEO. GAILEY CHAMBERS.

At the end of the first semester of the academic year 1916–1917, there were 22,912 grades assigned by the instructing staff in the University of Pennsylvania to the students in the undergraduate departments, that is, the students in the College, the Towne Scientific School, the Wharton School and the School of Education. There were five grades used, namely, D (Distinguished), G (Good), P (Passed), N (Not passed) and F (Failed). The significance of these grades, considered as percentages, varied in the different departments of instruction. There was no attempt at uniformity in this respect, nor was there any effort at uniformity in the percentage of D grades, G grades, etc.

The percentages of the different grades were as follows:

D's	14
G's	35
P's	38
N's	9
F's	3

There seems to be justification for assuming that each of these grades is the result of a measurement of some common trait, to which the term general ability might wisely be applied.

* Read before the meeting of the Association of Teachers of Mathematics of the Middle States and Maryland, December 1, 1917.

It is true that each measurement was a very complicated process beginning at the first session of the class at the opening of the semester, and continuing until the term examination was given at the end of the semester. It is also true that the process varied very greatly in different cases. For example, in some cases, the students whose progress was measured were seniors, and in others they were freshmen. Also, for example, in some cases the subject-matter of a technical laboratory course in electrical engineering was used in the measuring process; while in other cases, the subject-matter of a course in business law was used.

Nevertheless I believe we may assume that these 22,912 measurements were distributed according to the familiar normal probability curve as expressed by the equation

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}},$$

σ being the standard deviation for the whole set of measurements, x being the value of an individual measurement, and y the corresponding frequency of that particular measurement.

An essential condition for the application of the theory of probability to any set of statistics is that there shall be operating "numerous independent causes." Surely that is the case here.

Moreover, each group of students taken together in one application of the measuring process is approximately homogeneous in its stage of educational progress; that is, each section or class brought together for instruction is approximately homogeneous, at least in comparison with the crowd that gathers on the street when the fire bell rings, for example. Also the subject-matter and methods of presentation are adjusted to the stage of educational progress attained by the group.

Furthermore, speaking generally, we know of no cause that would tend to skew the distribution of the measurements in one of these instruction groups from the normal distribution. It is possible, however, that the standard deviation would vary with different groups, whereas, the assumption that the normal law of errors holds for the distribution of the whole set of 22,912 measurements implies the assumption that the standard deviation is the same for all of the instruction groups.

Therefore the results obtained from the use of the probability curve in such a case must be considered as very rough approximations. Nevertheless, I believe the considerations just given justify its use. The results are only slightly less reliable than are the grades themselves, and every college professor feels that much reliance can be placed on the grades he gives, and deans and executive committees feel sufficient reliance upon them to justify their dropping students from college, thus stopping their educational career along one line at least.

The area under the frequency curve will then represent the 22,912 grades. A table like that on page 219 in Thorndike's "Introduction to the Theory of Mental and Social Measurements" shows that the ordinates corresponding to $x = -1.83\sigma$, -1.15σ, 0.01σ, and 1.06σ will divide the area into portions corresponding respectively to the percentages of F's, N's, P's, G's and D's.

Also the ordinates corresponding to $x = -2.12\sigma$, -1.41σ, -0.49σ, 0.47σ and 1.46σ will divide the areas for the F's, N's, P's, G's and D's into equal parts respectively.

By taking a new origin 3σ to the left, a new unit equal to one tenth of σ and dropping the right-hand figure so as to be within the limits of accuracy of our data, we have the following as average values of the grades: F = 9, N = 16, P = 25, G = 35 and D = 45.

This computation was carried out to obtain the best values of the grades to use in ranking the undergraduate students, either all together or by groups. I applied these results first to the group of freshmen who entered directly from their secondary schools; that is, with less than a year of time between the completion of their school work and the beginning of their university work. There were 680 such freshmen.

In determining the values of the grades I decided to use the percentages of grades assigned to the whole undergraduate student body, rather than the percentages assigned to any smaller group. The corresponding percentages assigned to the group of freshmen just mentioned were as follows:

D's	12
G's	34
P's	39
N's	10
F's	5

If these percentages had been used to determine the values of the grades and if the origin had been properly chosen, we would have had the same average values for the grades. I believe now that it would have been better to have used only the grades of the freshmen, on the assumption that the common trait whose measure is represented by the grades would be approximately the same in amount for the freshmen, but would be higher for upper classmen. Fortunately, as it happened, the use made of these results is valid whichever set of grades is considered as the basis for determining the values of the respective grades.

Having these values of the grades and having the percentages of the different grades obtained by the individual students, we computed a rank number for each freshman. For example, a certain student's record showed 33 per cent. G's, 56 per cent. P's and 11 per cent. N's. His rank number was 274. The rank numbers would run from 90 to 450. They were computed by multiplying the percentages of the respective grades by the values of the corresponding grades, dropping the right-hand figure, and adding the results.

The whole set of students was then arranged according to the rank numbers, and separated into three groups: first, those who were in the upper quarter of the arrangement; second, those in the lower quarter; and third, those between the first two groups. Each student was then given a rank letter, U, M or L, according to whether he was in the upper, middle or lower group.

It would evidently be unwise to use the rank numbers as an accurate determination of individual ranks, because the data upon which the computation is based would not justify such precise results. I believe, however, that the rank letters, L, M and U, are sufficiently reliable for many purposes.

If now we assume that the 680 students whose ranking we have determined are distributed according to the normal law of frequency, then a table of values of the probability integral shows that the average values of L, M and U may be taken as 9, 20 and 31 respectively, with a fair degree of approximation.

Now take any group of students out of the whole set; for example, those who offered Latin for entrance; determine the percentages of those students that ranked in the L, M and U

groups respectively, and multiply those percentages by the corresponding values of those ranks and add the products, first dropping the right-hand figure to keep within the range of accuracy of the data. If the sum is more than 200 the group stands higher than a normal group, but if the sum is less than 200 the group stands lower than normal. The following tables give illustrations of such group standings.

RANKING OF GROUPS OF STUDENTS WHO ENTERED THE UNIVERSITY OF PENNSYLVANIA IN SEPTEMBER, 1916, DIRECTLY FROM SECONDARY SCHOOLS, BASED ON GRADES ASSIGNED IN ALL SUBJECTS AT END OF FIRST SEMESTER.

1. *Grouped According to Amount of Foreign Language Credited on Entrance.*

Amt. Lang.		No. with Rank			Total.	Amt. Lang.	No. with Rank			Total.	Amt. Lang.	No. with Rank			Total.
		L.	M.	U.			L.	M.	U.			L.	M.	U.	
2 units	No.	25	41	13	79	5 units	30	41	16	87	8 units	1	4	4	9
	Per cent.	32	52	16			35	47	18			11	45	44	
	Pt.	29	104	50	183		31	94	56	181		10	90	136	236
3 units	No.	25	48	14	87	6 units	27	64	46	137	9 units	2	5	8	15
	Per cent.	29	55	16			20	47	33			13	34	53	
	Pt.	26	110	50	186		18	94	102	214		12	68	164	244
4 units	No.	42	82	26	150	7 units	13	58	43	114	11 units	0	1	1	2
	Per cent.	28	55	17			11	51	38			0	50	50	
	Pt.	25	110	53	188		10	102	118	230		0	100	155	255

2. *Grouped According to the Number of Foreign Languages Credited on Entrance.*

Number of Languages.		No. with Rank			Total.
		L.	M.	U.	
1.....	No.	63	123	36	222
	Per cent.	28	56	16	
	Pt.	25	112	50	187
2.....	No.	94	203	117	414
	Per cent.	23	49	28	
	Pt.	21	98	87	206
3.....	No.	8	18	18	44
	Per cent.	18	41	41	
	Pt.	16	82	127	225

3. *Grouped According to the Language Credited on Entrance.*

Language.		No. with Rank			Total.	Language.	No. with Rank.			Total.
		L.	M.	U.			L.	M.	U.	
Latin.....	No.	110	238	130	478	German.....	122	232	127	481
	Per cent.	23	50	27			25	48	27	
	Pt.	21	100	84	205		23	96	84	203
Greek	No.	1	20	18	39	Spanish	1	7	5	13
	Per cent.	3	51	46			8	54	38	
	Pt.	3	102	143	248		7	108	118	233
French.....	No.	41	86	44	171	No Latin....	55	106	41	202
	Per cent.	24	50	26			27	52	20	
	Pt.	22	100	81	203		24	104	62	190

The first number in the column headed "Total" gives the number of students entering with the corresponding offering in Foreign Language as given in the left-hand column.

The second number in the "Total" column gives the standing of the group. The larger the number the higher the standing. The lowest possible standing would be 90 and the highest 310.

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA.

THE PLACE OF ARITHMETIC IN THE HIGH-SCHOOL CURRICULUM.

By William Wiener.

A comparative study of methods of school administration and plans upon which the writer has for some time been engaged has left the impression that courses of study in high schools smack too much of similarity. The good school becomes the object of imitation for other institutions and thus there are frequently and thoughtlessly repeated the foibles and weaknesses of strong leaders. This similarity is very apparent in the syllabi and courses of study in mathematics in the high schools. Such a condition has been brought about because it is less burdensome, following the line of least resistance, to copy study-courses, and to have the high schools patterned the one after the other, than to be original and thoughtful as to the social function of the high school. Arithmetic has not been in fashion for high schools, and it has been tabooed as a high-school subject, the thought being that arithmetic or the teaching of arithmetic is beneath the dignity of the high-school instructor. Pupils have unwittingly acquired this feeling of indifference towards arithmetic and its processes, because of the teachers' attitude toward the subject. Despite the fact that a working knowledge of arithmetic has been considered out of fashion by teachers and pupils, boys and girls of the adolescent age, with their appreciation of real values, can be induced to have an extremely high regard for this subject because it functions into their own lives and also functions into world activities and relations. The high-school period is the one in which the boy and girl is ever ready to work at arithmetical problems, feeling that experience in their solution and handling affords a reserve force and power for future advance and effort. Therefore the high-school youth are easily brought, when once the importance of the subject has been demonstrated, to look upon arithmetic as a most necessary part of the high-school career. The adolescent period, the period of

early educational life of the child, is a time of value appreciations. Is it not a fact that the emphasis placed upon the exploitation of the three R's is a naturally intuitive one? Reading must of necessity have been the first means of obtaining quiet thought transference, through records. Writing is the next method of gaining thought transference by making records. A knowledge of arithmetic, on the other hand, brings about the summation of ideals and experiences. For life is full of arithmetical situations and problems which the simple processes of arithmetic help to clarify. The school must prepare for the life to be lived. If the school through its teachings fails to make that life efficient and comfortable for the individual by developing and training along those lines which make possible the greatest rounded efficiency, the school does not perform its full social duty. A knowledge of arithmetical processes and their applications is the daily need of the individual members of society. Existence of course may continue without this knowledge, but individual social efficiency is to a great extent diminished when this arithmetical information and experience has been neglected.

It is almost unnecessary to mention the fact that in our daily walks of life systematic applications of the principles of arithmetic do necessarily produce habits of thrift and value-appreciations, an essential characteristic of American life, a characteristic necessary for the successful continuance of any nation. Naturally the impressionable period of adolescence is that during which the importance of this branch of knowledge can be most firmly established. No pupil can afford to be without training in arithmetical processes and analyses. It matters not whether the pupil intends going to college, or into business, or into the home. Arithmetical training is fundamentally and absolutely a necessary factor in a rounded training. It provides the tools for use in higher mathematical work by training for accuracy and complete conceptions of number processes, combinations and problems.

Unless the accuracy habit is fixed early in the child mind, the success of the child with mathematical subjects is ofttimes made impossible. Proper training in arithmetic therefore stresses the fundamental processes of mathematics as well as the reasoning powers of the child and thus furnishes mental attitudes and apti-

tudes, which insure a great likelihood of success in dealing with higher or advanced mathematical subjects. It need not be mentioned to you who till and toil in the mathematical field that you will reap the greatest harvest of mathematical successes whenever the mathematical soil of youth furnished you has had the advantage of previous treatment with regenerating and fertilizing mathematical experiences. It has been noted in our own school that where arithmetic has formed the basis of the mathematical superstructure of the students' course of study, failures in algebra, geometry and higher and applied mathematics are reduced to an absolute minimum. Our records definitely prove this statement.

Let us note further what the world requires. It demands not only a knowledge of correct methods but insists upon an ability to handle figures and arithmetical knowledge with precision and accuracy. The old notion of crediting the pupil with approximate or inaccurate results, as practised in the elementary grades, does not appeal to the business world, which has no idea of permitting inaccurate answers. This practise destroys the appreciation of the value and need for arithmetical accuracy in results. Here is originated the vicious and demoralizing mental habit that disregards the advantage of accurate work, in spite of the social and commercial demand for it. The high school is the place where this false impression, that inaccurate results are of value, must be destroyed. I can see no reason for the continuance of the practise even in the grades. Accuracy and precision with simple processes are absolutely required by business interests. Can a business house credit or even tolerate inaccurate results, and continue in operation? What would become of any firm that permitted such ridiculous practises? Check systems of different kinds, to insure efficiency in business affairs, are being constantly devised by accountants and auditors. The habit of accuracy is therefore a business ideal and a valuable asset.

In the elementary schools arithmetic is taught by teachers who deal with various parts of the subject, as directed in the syllabi of the school. The pupil is therefore taught arithmetic by divisions of the subject and made to feel that each division is a separate, independent and new topic. That this is the ex-

perience of boys and girls we can recall in our own time as well as observe at the present time. To be specific, long division is of a different genus from short division, entirely unrelated the one to the other, or to any other arithmetical subject. It remains for the high school to show the unity of arithmetical processes and thought, by co-ordinating and correlating these desultory elementary-school experiences with individual arithmetical topics. It has been the experience in our high school that where commercial, technical or other pupils have been thoroughly grounded in arithmetical fundamentals, it is possible for them to take advanced courses in mathematics with a larger measure of success and understanding than is possible for pupils who have not had such arithmetical preparation. The percentages of promotion of the arithmetically prepared pupils in algebra and geometry and applied mathematics are far above the average of those who were not so prepared. It is a further interesting fact to note that the boys and girls who have had arithmetic in the high schools and later entered colleges have stood the acid test of criticism and experience with college mathematics more creditably than have those who had not had arithmetic in the high school.

Arithmetic must be considered a vital and fundamental part of the high-school curriculum, despite the very general contention that it *is taught* in the elementary schools. Pupils and teachers must be made to feel that the public demand for arithmetic is based on social necessity. The high school therefore must attack this subject with thoroughness and interest, impressing upon all the importance of the accuracy and precision habit, as one of the educational factors bound to insure the success of the individual at college, in business and in the home.

CENTRAL HIGH SCHOOL,
NEWARK, N. J.

VOCATIONAL COURSES IN MATHEMATICS FOR SECONDARY SCHOOLS.*

By David Eugene Smith.

Professor Smith stated that it was his intention to speak rather upon the general problem than upon any details. He stated that the problem of the industrial mathematics was a new one in all new countries, but an old one in the older civilizations. The reason why the problem is new here is that a country like ours tends to consider the individual more than the state, whereas in various older countries the opposite is the case. All types of vocational training have been thought to arrest the development of the individual, and this has actually been the case in certain types of schools in Europe.

Our problem, he said, is to avoid this danger and at the same time to secure the good results that must come to many students from this type of education. He stated that in the older countries the tradition still obtained to a considerable extent that the child followed the trade of the parent, and that it was therefore natural to expect that he should be put into a vocational school preparing him for that trade. This was favored by governments on the well-known ground that a generally educated proletariat was not always desirable.

Professor Smith then took up the influences of the world war upon this problem. It was his opinion that the poverty that would result in all countries would make it necessary to have a much better type of industrial education than we have had in the past. Individualism has shown unexpected weakness in certain respects and paternalism has shown unexpected strength, so that possibly our views as to arresting the development of the individual may be subject to change. He also stated that our general education, which has been largely in the hands of the

* Report of remarks by Professor Smith at a meeting of the New York Section of the Association of Teachers of Mathematics in the Middle States and Maryland, Oct. 26, 1917.

so-called commercial middle class, is likely to pass to quite an extent into the hands of labor and that he felt that the change would not be undesirable. It would result in a more economic and productive education. It was his opinion that the new order would be entirely opposed to a narrow education for the reason that labor would not block the progress of its children.

He felt that labor would demand from a school its money's worth, and that what Dr. McFarlane has called the camouflage of pedagogy would no longer be in the ascendency. Women, having shown a great power recently in all industrial lines, would be educated for industry in a way that was not thought of when the war began.

In order to arrange for a type of student suited to a particular type of school, he stated that there was considerable hope of help from the psychologists, although at present this movement was merely in its infancy.

He then summarized the work that was being done in other countries in the matter of industrial education and drew the conclusion that for America the best grade in which to seriously begin this work was the tenth.

He advocated a general reform of our mathematical work before the tenth grade, with the result of having all pupils know the useful part of algebra, the significance of intuitive geometry, the nature and purpose of trigonometry, and the meaning of a mathematical proof. He felt that after that they might safely be allowed to specialize in industrial and commercial lines, or in the type of pure mathematics now in our secondary schools.

He called attention to the two extremes in the teaching of industrial mathematics, the first being that of laying a thorough scientific basis in pure mathematics and building the applied mathematics upon this foundation, and the second being the Perry idea of beginning at once with applied mathematics and taking only so much of the theory as might be necessary for a practical working knowledge.

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PRACTICAL MATHEMATICS FOR HIGH SCHOOLS.

By W. H. Dooley.

One of the most constructive criticisms that have been made of secondary-school mathematics, by a general educator, was made by H. C. Morrison, State Commissioner of Education, of New Hampshire, when he said, "The traditional courses in algebra, geometry, trigonometry and advanced algebra must be revised as to organization and content to meet the need of the adolescent, and the social purpose of the high school . . . mathematics must be considered as a language to interpret science. . . . It must be presented to give immediate opportunity for functioning. It should be adapted to meet the needs of the different courses." These suggestions were offered after years of observation of high-school teaching of mathematics, and were no doubt due to the following causes: change in type of pupil in the high school; feeling that the industrial and commercial needs should be recognized in secondary-school mathematics; disbelief in the formal discipline theory.

The thoughtful student of education will ask why are complaints made about the subject of mathematics that has been in our course of study as a compulsory subject during the last century? An investigation will show that the high school was established primarily for a group of pupils who had intellectual taste—ability to pursue mathematics, foreign languages, history and science. During the last twenty years the high-school population has increased by leaps and bounds, out of proportion to the increase in population. For example, in the year 1899–1900, there were in the public high schools of the United States about 221,000 students, or about .36 of one per cent. of the population. In 1908–1909, these same public high schools had 863,000 students, or about one per cent. of the population. These figures show that within ten years the high-school population increased 300 per cent. The high schools were once patronized by the children of the professional classes, and especially those children

who intended to enter the higher schools of learning, but the above statistics show that this is no longer true. In the minds of the people the high schools are becoming more popular, and there is a constantly increasing disposition to send children to high school in order to give them two or more years of education.

It is clear that the high school of to-day has children of all classes; those of the intellectual, the artistic and the practical type.

While the children who have a special genius or talent for the formal or abstract studies have been able to grasp the traditional course in mathematics, so as to pass the promotional tests, they have received little if any training in the application of mathematics to the trades and the home. They have graduated from the course in mathematics without being able to commercialize or apply the same to life. The other types of mentality have neither the interest nor the mental equipment for abstract mathematics, and invariably fail to meet the promotional tests. This accounts in some measure for the large percentage of failures.

If the modern high school expects to have efficient courses, it must supplement and rearrange its program in mathematics so as to meet the interests and mental equipment of all the boys and girls. This means that the school must teach every kind of mathematics which society employs in the conduct of its affairs; not only to encourage the study of each, but to give to every variety of mind that interest and growth in mathematics which is necessary to power and self-confidence. In attempting to do this, let us not destroy or weaken the effective work that is being done in pure mathematics for the child of intellectual interests.

The movement to adjust mathematics to the capacities and the practical needs of the pupils has already made some progress in the upper grades. The essentials of arithmetic are taught within the first six years of the elementary school. The last two years are devoted to a study of community business arithmetic, that is, a study of the application of arithmetic to simple problems. There is no logical reason why this movement should not extend to the first year and possibly for some pupils to the second year of the high school. Some teachers may feel that vocational

mathematics should follow, rather than precede, the regular courses in algebra and geometry. The intellectual type of child may well take it, after the theory, if the same does not interfere with the preparatory work for college which is usually carried on very efficiently the last two years. For the other types of pupils, the vocational mathematics should be given during the first two years, otherwise the great majority who fail to complete over two years' work will leave school without this training.

Modern psychology teaches us that the attitude of the mind in learning mathematical principles is different from that utilized in applying the principles to situations; therefore a course in mathematics in a high school should include an effective training in the applications of principles in mathematics to the trades and industries that are common to every community. The majority of pupils will be obliged to earn their living in either the productive or distributing branches of these trades.

The question may be asked, What are the trades and industries common to every industry? For boys—wood-working, metal-working (including machine shop and sheet-metal work), plumbing, power-plants (including steam and electrical work) and the building trades. Oftentimes the community may have a single industry, such as textiles, tanning, boot and shoe manufacture and agriculture. The mathematics in use in these industries should also be taught. For girls—the true vocation of all women, homemaking, trained nursing, dressmaking, millinery, clerical work, etc.

The next question that may be asked is, How can the average teacher prepare himself or herself to teach the mathematics underlying these trades? The experience of the writer has been that this may be done in a very short time. The first step is for the teacher to visit the trade and make a collection of materials, hand-tools and illustrations of power-tools that involve mathematics. The teacher will soon acquire an appreciative knowledge of trade conditions and the types of problems used in each trade. With a well-graded text-book in the subject, the teacher will be equipped to handle a class.

The first part of every new subject should be devoted to a discussion of the subject-matter. Find out what the pupils already know, and then show the value of more knowledge on

the subject. As far as possible the teacher should present the subject objectively, laying great emphasis on practical proofs (demonstrations) of mathematical principles. For example, in teaching the relation between the diameter of a pulley and the distance around it, ask a pupil to draw a chalk-line on the floor and then mark the face of the pulley at the beginning of the chalk-line. Then roll the pulley along the chalk-line for one complete revolution. The diameter of the pulley is the longest distance across the pulley. Dividing the distance around the pulley by the diameter will give the ratio of the diameter to the circumference—the constant π.

The keynote of to-day is thrift or conservation. The habit of thrift can be cultivated to a great degree through proper correlation of the home and school. The problems of the home may be made the basis of practical mathematics in school. Problems in homemaking should include distribution of income, based not only upon the problems of the text-book, but on the data obtained by members of the class. Each girl should keep to criticize the disbursements expressed in per cent. of the income in it. It should be balanced monthly and the pupil made to criticize the disbursement expressed in per cent. of the income. Constructive comments may be made by the teacher and other pupils, showing where improvements may be made. When mathematics of the home are well taught by an interesting teacher, it not only helps to introduce into the homes that methodical spirit which regulates expenses by receipts, and makes the product of thrift fruitful, which we need so much to-day, but develops an interest in the field of mathematics and shows that the study of this subject is worth while. Similar problems should be presented dealing with the vocational activities of women. Remember that all problems in vocational mathematics should be made to arise in a vital and natural way, so that the pupil will recognize the need for the training in the principles of mathematics which the teacher requires.

NAVY YARD,
BROOKLYN, N. Y.

IS THE PRESENT ENTRANCE REQUIREMENT IN ALGEBRA EXCESSIVE IN AMOUNT? DOES IT EXPECT TOO GREAT MATURITY OF THE STUDENT?

By Elizabeth Forrest Johnson.

That the present algebra requirement has increased very markedly in actuality if not in the actual words in which it is described during the last fifteen years, I think is unquestionably true. From this increase, a very marked increase, an increase in difficulty of requirement, an increase in mathematical grasp expected of our students—from this increase I think there comes a very great temptation to the secondary schools to shave the requirement down to its lowest terms, to plan very closely, to economize at every point, to spare the students, to help the students out. And then in the last year when the students are finishing the requirement, we face the question, "Is it not perhaps altogether excessive? Doesn't it assume a maturity that our students have not?" My answer to that question sounds paradoxical when it is stated, and yet I very firmly believe it is true. My answer is something like this: If the algebra requirement of the colleges is interpreted grudgingly, if a school in planning its course holds itself down to the requirement reduced to lowest terms, the course will be narrow and barren, breathless, and of too great pressure; whereas, if the requirement is interpreted generously, if the course gives not merely what is required, but a good deal more than is required, then the course will prove to be not too taxing.

For consideration to-day, I wish to speak only of the requirement as defined by Bryn Mawr College, and as defined by the College Entrance Examination Board, because those two requirements taken together seem rather thoroughly to cover the field and because they are the requirements with which I am most familiar.

Those requirements both include elementary operations, frac-

tions, factoring, radicals, exponents—integral, fractional, positive, negative—quadratics, problems, ratio and proportion, arithmetic and geometric progressions, the binominal theorem for positive integral exponents. The College Entrance Board requires, in addition, familiarity with the use of graphical methods and illustrations, and ability to solve problems chosen from mensuration, physical formulas, and the like. Bryn Mawr College does not require these, but does require ability to give rigorous mathematical proofs of certain pieces of algebraic theory—the remainder theorem, the factor theorem, for example—the proof of the binomial theorem, the proof that the quadratic equation has two and only two roots; proof connected with the theory of quadratics, proofs of the meaning of the fractional and negative exponents, the derivation of the formulas used in the progressions—twenty or more of such proofs, and all pretty difficult for the average student.

Any mathematician reading those requirements in the light of the papers set by Bryn Mawr College, and by the College Entrance Examination Board, will feel, I think, that, where they diverge, the Bryn Mawr requirement has the forward look to pure mathematics, and the College Board requirement has the forward look to applied mathematics. Now, my conviction, very steadily deepening from year to year, is that a mathematics course which simultaneously meets both these requirements is an almost ideal course, and that it is not too heavy nor expecting too great maturity if certain rather definite conditions can be established in the course. I sum up these conditions somewhat in this way:

In the first place, the mathematics course of the school must be planned as a whole, and from the first grade through the twelfth, by mathematicians.

In the second place, the course should include certain items, although, so far as I know, no college requirement specifically mentions them, or refers to them. In the lower grades, before the secondary school, the course should include geometric applications; for instance, actual measurements, some work in inductive geometry. The course should include the use of algebraic symbols, at least as early as the sixth grade. And there should be careful drill to train in speed and accuracy, such drill

being based on some such devices as the Courtis tests. In the secondary school, the graph should be used to introduce simultaneous equations, not left until the very end of the course and dragged in as a long and cumbersome application to the simultaneous equation already familiar to the students. And somewhere in the secondary school there should be included a good course in inductive geometry.

In the third place, time enough should be given in the secondary school to do this mathematics course; that is, at least four years of work in the secondary school, not, I think, at all necessarily five times a week. Perhaps the ratio should be about two years and a half for algebra to a year and a half for geometry, but throughout the course there should be an abundance of geometric application to algebra, and of algebraic application to geometry.

In the fourth place, if the mathematics department consists of more than one teacher, at least one teacher should be a mathematician whose forward look is naturally to pure mathematics; and at least one whose forward look is naturally to applied mathematics.

A course planned thus generously along these lines will, I am sure, bring results that are well worth the time given it. Facility in handling the tools of mathematics, skill and power in abstract reasoning of a really difficult sort, developed by the theoretical work mentioned before; power to apply later these skills and methods to the natural sciences—that is what a school course preparing for these two requirements will give, if the schools will give the mathematicians a chance by allowing sufficient time for the work and by paying proper attention to its planning. If the mathematics course aims at less than this, it aims, it seems to me, at very much less. Either, it will aim merely at preparation for such requirement as the College Entrance Board requirement—a requirement utilitarian, scientific, sound, but from the point of view of the mathematician lacking the vision, the forward look towards higher, pure mathematics that those rigorous proofs demanded by Bryn Mawr College give our students, a glimpse that they enjoy even while they find them difficult; or, it will aim merely at meeting a requirement such as the Bryn Mawr requirement—beautiful mathematics, but with-

out that connection with the natural sciences that, for so many young minds of the concrete type, makes all the difference between light and darkness. To accomplish either aim without the other is, paradoxically enough, to work too hard and attain too little; to do both is practicable, economical of time and energy.

Specifically, I should advocate omitting certain definite items from the requirement, not that to do so will save much time, but because I think mathematically they are out of place. I should omit the proof, not the applications, of the binomial theorem, for I think that proof is best given once for all in the college course in connection with permutations and combinations. I should omit, possibly, one or two other definite items of proof. But the omission or inclusion of these is to my mind really unimportant. The important matter is the generous scope of the planning of the mathematics course, and the planning of it as a mathematical unity from the first grade to the twelfth grade.

THE BALDWIN SCHOOL,
BRYN MAWR, PA.

THE DUTY OF THE EMPLOYER IN THE RECONSTRUCTION OF THE CRIPPLED SOLDIER.

By Douglas C. McMurtrie,

Director Red Cross Institute for Crippled and Disabled Men, New York City.

We must count on the return from the front of thousands of crippled soldiers. We must plan to give them on their return the best possible chance for the future.

Dependence cannot be placed on monetary compensation in the form of a pension, for in the past the pension system has proved a distinct failure in so far as constructive ends are involved. The pension has never been enough to support in decency the average disabled soldier, but it has been just large enough to act as an incentive to idleness and semi-dependence on relatives or friends.

The only compensation of real value for physical disability is rehabilitation for self-support. Make a man again capable of earning his own living and the chief burden of his handicap drops away. Occupation is, further, the only means for making him happy and contented.

Soon after the outbreak of hostilities the European countries began the establishment of vocational training schools for the rehabilitation of disabled soldiers. They had both the humanitarian aim of restoring crippled men to the greatest possible degree and the economic aim of sparing the community the burden of unproductivity on the part of thousands of its best citizens. The movement had its inception with Mayor Édouard Herriot of the city of Lyons, France, who found it difficult to reconcile the desperate need for labor in the factories and munition works while men who had lost an arm or a leg but were otherwise strong and well were idling their time in the public squares. He therefore induced the municipal council to open an industrial school for war cripples which has proved the example and inspi-

ration for hundreds of similar schools since founded throughout France, Italy, Germany, Great Britain and Canada.

The disability of some crippled soldiers is no bar to returning to their former trade, but the injuries of many disqualify them from pursuing again their past occupation. The schools of training prepare these men for some work in which their physical handicap will not materially interfere with their production.

The education of the adult is made up largely of his working experience. The groundwork of training in his past occupation must under no circumstances be abandoned. The new trade must be related to the former one or be, perhaps, an extension or specialization of it. For example, a man who had done manual work in the building trades may by instruction in architectural drafting and the interpretation of plans be fitted for a foreman's job, in which the lack of an arm would not prove of serious handicap. A trainman who had lost a leg might wisely be prepared as a telegrapher, so that he could go back to railroad work, with the practice of which he is already familiar.

Whatever training is given must be thorough, for an adult cannot be sent out to employment on the same basis as a boy apprentice. He must be adequately prepared for the work he is to undertake.

The one-armed soldier is equipped with working appliances which have supplanted the old familiar artificial limb. The new appliances are designed with a practical aim only in view; they vary according to the trade in which the individual is to engage. For example, the appliance for a machinist would be quite different from that with which a wood turner would be provided. Some appliances have attached to the stump a chuck in which various tools or hooks can interchangeably be held. The wearer uses these devices only while at work; for evenings and holidays he is provided with a " dress arm " which is made in imitation of the lost natural member.

An important factor in the success of re-educational work is an early start, so that the disabled man shall have no chance to go out unemployed into the community. In even a short period of exposure to the sentimental sympathy of family and friends, his " will to work " is so broken down that it becomes difficult again to restore him to a stand of independence and ambition. For this

reason, therefore, the plan for his future is made at as early a date as his physical condition admits, and training is actually under way before the patient is out of the hospital.

In the readjustment of the crippled soldier to civilian life, his placement in employment is a matter of the greatest moment. In this field the employer has a very definite responsibility.

But the employer's duty is not entirely obvious. It is, on the contrary, almost diametrically opposite to what one might superficially infer it to be. The duty is not to "take care of," from patriotic motives, a given number of disabled men, finding for them any odd jobs which are available, and putting the ex-soldiers in them without much regard to whether they can earn the wages paid or not.

Yet this method is all to common. A local committee of employers will deliberate about as follows: "Here are a dozen crippled soldiers for whom we must find jobs. Jones, you have a large factory; you should be able to take care of six of them. Brown, can you not find places for four of them in your warehouse? And Smith, you ought to place at least a couple in your store."

Such a procedure cannot have other than pernicious results. In the first years of war the spirit of patriotism runs high, but experience has shown that men placed on this basis alone find themselves out of a job after the war has been over several years, or in fact, after it has been in progress for a considerable period of time.

A second weakness in this method is that a man who is patronized by giving him a charity job comes to expect as a right such semi-gratuitous support. Such a situation breaks down rather than builds up character, and makes the man progressively a weaker rather than a stronger member of the community. We must not do our returned men such injury.

The third difficulty is that such a system does not take into account the man's future. Casual placement means employment either in a make-shift job as watchman or elevator operator such as we should certainly not offer our disabled men except as a last resort—or in a job beyond the man, one in which, on the cold-blooded considerations of product and wages, he cannot hold his own. Jobs of the first type have for the worker a future of

monotony and discouragement. Jobs of the second type are frequently disastrous, for in them a man, instead of becoming steadily more competent and building up confidence in himself, stands still as regards improvement and loses confidence every day. When he is dropped or goes to some other employment, the job will have had for him no permanent benefit.

Twelve men sent to twelve jobs may all be seriously misplaced, while the same twelve placed with thought and wisdom and differently assigned to the same twelve jobs may be ideally located. If normal workers require expert and careful placement, crippled candidates for employment require it even more.

The positive aspect of the employer's duty is to find for the disabled man a constructive job which he can hold on the basis of competency alone. In such a job he can be self-respecting, be happy, and look forward to a future. This is the definite patriotic duty. It is not so easy of execution as telling a superintendent to take care of four men, but there is infinitely more satisfaction to the employer in the results, and infinitely greater advantage to the employee. And it is entirely practical, even in dealing with seriously disabled men.

A cripple is only debarred by his disability from performing certain operations. In the operations which he can perform, the disabled man will be just as efficient as his non-handicapped colleague, or more so. In the multiplicity of modern industrial processes it is entirely possible to find jobs not requiring the operations from which any given type of cripples are debarred. For such jobs as they can fill the cripple should be given preference.

Thousands of cripples are now holding important jobs in the industrial world. But they are men of exceptional character and initiative and have, in general, made their way in spite of employers rather than because of them. Too many employers are ready to give the cripple alms, but not willing to expend the thought necessary to place him in a suitable job. This attitude has helped to make many cripples dependent. With our new responsibilities to the men disabled in fighting for us, the point of view must certainly be changed. What some cripples have done, other cripples can do—if only given an even chance.

The industrial cripple should be considered as well as the

military cripple, for in these days of national demand for the greatest possible output there should not be left idle any men who can be made into productive workers.

With thoughtful placement effort, many men can be employed directly on the basis of their past experience. With the disabled soldiers who profit by the training facilities the government will provide, the task should be even easier.

This, then, constitutes the charge of patriotic duty upon the employer:

To study the jobs under his jurisdiction to determine what ones might be satisfactorily held by cripples. To give the cripples preference for these jobs. To consider thoughtfully the applications of disabled men for employment, bearing in mind the importance of utilizing to as great an extent as possible labor which would otherwise be unproductive. To do the returned soldier the honor of offering him real employment, rather than proffering him the ignominy of a charity job.

If the employer will do this, it will be a great factor in making the complete elimination of the dependent cripple a real and inspiring possibility.

MATHEMATICS OF FINANCIAL PROBLEMS.

By L. M. Webster.

Dr. Davenport, of the University of Illinois, says: "We have entered upon an era of universal education which means the education of all sorts of people for all sort of purposes. It must not only fit for the so-called learned professions, it must also train for the commercial and industrial world." Dr. Davenport has certainly struck the keynote of to-day's topic. Most teachers admit that the pendulum has swung too far to the cultural side and that we must introduce the commercial and industrial factors, not to the exclusion of the cultural; it is possible to have our work broadened to the exclusion of no element. We can even emphasize the cultural by bringing the laws of higher mathematics into effectual contact with the daily routine of the mercantile house. Great colleges and universities record with pride the achievements of their graduates. This is but a recognition that the highest product of their training is the disciplined mind at work. While "experience is the best teacher" is an adage as true as it is old, statistics show that a mind trained for the special line of work to be undertaken is a valuable asset. Acknowledging that the Perry plan has some advantages, I still hold that the worker in mathematical fields, at least, does more efficient work in the ratio of his understanding of the principles he is applying. It is only a few years ago that this was recognized by the establishment of commercial high schools and courses in colleges and universities dealing even indirectly with the problems of great corporations. The value of these courses has been demonstrated but we must look to the teacher of financial problems to so standardize his work that college faculties will allow its full measure of credit. The war has called thousands. Their posts are being filled by graduates and undergraduates who have been obliged to assume responsibility with more meagre equipment than their older brothers. I understand that in one New York City high school the registration is 1,000 less than it was a year ago. Are these boys ready for the

work they are undertaking? Have they been so trained that they can apply their mathematics to the problems which will confront them? They have probably taken two years each of algebra and geometry; perhaps trigonometry or solid geometry if they have been preparing for college; have they learned in the algebra course that computation of compound interest is a practical application of geometric series, that present worth is an illustration of negative exponents? Can they use the laws of logarithms to express the conditions they will meet in the amortization of interest-bearing debts? Do they hold the tool of the successive discount formula, can they calculate the present worth of a perpetuity, the value of a foreborne annuity, or even the annuity that a saving of $1 per week will purchase. I believe that the place to derive these formulæ and emphasize their application to the financial world is the classroom. The time to master them is not when, as an employee, curiosity prompts investigation of tables. Each one of us would, indeed, feel guilty in slighting the Pythagorean formula. We apply it wherever and whenever we can from the day the student attacks the proposition to the hour he completes his calculus, but do we teach Euler's amortization equation? In comparing the two does not the latter afford ample opportunity for the mathematical gymnastics used in this discussion of series, exponential equations, and rates; and is it not of far greater value to the student who seeks a position in the financial world?

Can we not spend more time on graph work. All large corporations are using this mode of interpretation. Railroads seem to cling to the step method; and the straight-line graphs and the Ben Day rods are in universal favor. There seems to be no general satisfactory method of graphing three variables. As teachers of analytics we delight in showing our pupils why the parabola is called the square root conic and the hyperbola the constant product and it is on account of their practical value that we consider the cissoid and conchoid as milestones. In calculus we aim to reach as soon as possible maxima minima, or the discussion of rates. We are sure to use the catenary in our first lesson on hyperbolic functions; but are we using to their fullest value these tools, legacies of master minds, sharpened by their correlation with the other sciences and polished to an immeas-

urable fineness by their constant use in pure mathematical fields if we are silent about their value in the financial world?

I realize what the commercial high schools are doing. I have visited classrooms and enjoyed the work. Every problem had its practical side. My plea is for the application of college mathematics to the financial problems that are confronting us. It will not make our work any the less technical to include applications to bond issues and sinking funds. I think we would keep the same high standard and add a new dignity. We would establish a means of bringing abstract truths into effectual contact with daily business.

Columbia, New York University, Chicago University, and many others are offering courses in finance in extension departments. The Wall Street Division of New York University School of Commerce, Accounts and Finance is in its third year. Its growth measures the great development in our financial institutions, both as regards numbers and their methods of reaching the greatest efficiency. The close co-operation between education and the problems of the financial world is the keynote of the curriculum. The student body includes managers of financial departments and clerks seeking advancement. I have not been able to learn what proportion of the students are college graduates to whom a course in higher mathematics applied to financial problems would have appealed.

The business of the country to-day is war and the great trust and banking concerns are meeting the struggle against the odds of a greater volume of work, the loss of experienced workers, and the blunders of novices. Cannot the colleges help by supplying men and women who can enter the financial world armed with *practical* technique, those who as students have correlated higher mathematics with commercial law.

Let me give you a few figures. The New York City Clearing House passes annually $100,000,000,000. These clearings are plotted and distributed each week. In the United States there are about 200 clearing houses serving over 2,500 banks which employ over 10,000 in the capacity of experts in financial lines.

Again graphs of deposits, resources, commodity prices, absorption of capital, velocity of absorption of money and thousands of other conditions are published weekly. Many college

graduates have entered this field within the past three years and there is room for many more. That the business world is ready to co-operate with the colleges there is no doubt.

Statistics show that the college-bred man goes up the ladder with much more rapid strides than his fellow worker who climbs by experience. But, ought not a college graduate to have the equipment for an advanced position in the financial field just as the graduate of a technical school has for his life work? In other words, is not an intensive course in financial mathematics valuable? Is not to-day the time to standardize such a course?

The world is looking to institutions and individuals for help along many lines, not the least of which is efficiency in the financial field. After the great struggle is over the changes in education will no doubt be radical. Surely industrial mathematics will be emphasized, but the mathematics of finance will be a controlling factor.

HUNTER COLLEGE,
NEW YORK CITY.

"To be true to duty;
To maintain self-respect;
To keep memory unsullied by any wrong;
To be faithful to a friend, generous to a foe,
And upright in all the relations of life—
This is success."

REPORT OF THE COMMITTEE ON ELEMENTARY-SCHOOL MATHEMATICS, OF THE ASSOCIATION OF MATHEMATICS TEACHERS OF THE MIDDLE STATES AND MARYLAND.

Your Committee on Elementary-School Mathematics is again under the painful necessity of dashing to earth any hopes and expectations which may have arisen from the announcement in the program looking forward to a final report.

If there is any one thing which the work of this committee lacks, it is finality. The best we can do at this time is to present for your consideration certain tentative recommendations in the hope that still further light may be shed upon our difficult problem, and that the members of the Society may devote a modicum of time and attention to its consideration.

Let it be stated at once that we are not prepared to lay down a definite program or set forth an ideal curriculum. Our investigations have brought to light many widely varying opinions which were dependent on local conditions in general and upon administrative questions in particular. We feel that a syllabus, such as we might outline at this time, would be highly artificial and nowhere generally acceptable.

The trend of mathematical pedagogics for the past decade has taken two directions which, at first thought, seem contrary: these are (1) a demand for greater rigidity and clarity in definition and development of the fundamental processes. (2) A search for more numerous points of contact between the world of logic and the world of experience. The immediate result has been an apparent division of our forces into opposing camps. It should be noted at once that there is no really essential contradiction involved in these two distinct aims; provided only that the antique doctrine of immediate intuition of fundamentals is abandoned; a question which it is not within our province to consider. But what *is* inescapably involved is a careful and prolonged study on the part of teachers as to the relative degree of refinement of process suitable for classroom purposes under

varying conditions, and the relative extent to which, in our teaching, application should be made of mathematical processes to actual concrete facts. This matter is one which it is well nigh impossible to dogmatize about. The best solution seems to lie in the direction of a minimum syllabus of essentials, allowing the widest liberty of choice of applications, depending upon local conditions; and also giving due regard to individual and local differences in degree of recognition of the demands of logical accuracy. In reference to the last mentioned, it may well appear that the steps, in each demonstration or solution, form a *continuous* class, which is simply isomorphic with a concrete instance, and that no *proof* is really perfect as stated. Even the Euclidean processes of logic, strict as they are, are capable of further intensive development.

The moral of which is that we are not justified in expecting from our students either in elementary school or in high school that they attain a fixed standard of logical precision, Euclidean in geometry, or Robinsonian in arithmetic. It is the fear of violation of Euclidean (or Wentworthian) standards of process which has kept geometry out of the elementary schools. Teachers have failed to distinguish between the significance of error in conclusion and that of incompleteness in process. Both are regarded as delinquencies and equally culpable. At the outset, then, we may safely assert that *degrees of accuracy* and *degrees of completeness* should be recognized and carefully distinguished by teacher and pupil alike.

To illustrate: It should appear that the results of a financial transaction must be accurate to the last cent, but no further; but that to find correct to a ten thousandths of a square inch the amount of tin needed for a cylindrical bucket is highly absurd. So it is absurd to expect from a fourth-grade pupil any form of "proof" of the rule for division of fractions, or of a high-school pupil (except possibly in the fourth year) a "proof" by the so-called method of limits; but it is equally a source of discord between elementary-school and high-school mathematics for the elementary-school student to escape entirely the demand for fairly careful abstract reasoning.

The tendency in elementary schools for the past two decades seems to have been away from all abstractions. In their effort

to make the arithmetic *applicable*, the elementary-school authorities seem to have limited the arithmetic to consideration of concrete magnitude. This doubles the difficulty encountered by the student upon entering high school. In most cases, not only must he master a new notation, but he must also comprehend the significance of general truths in mathematics. What teacher has not found the statement of a troublesome principle made perfectly clear by the insertion of the word "dollars" at certain convenient points? Any abstract principle is, by its very nature, removed from the particular concrete setting which gave it birth. It requires to be memorized carefully, and its general character emphasized by a large variety of illustrations. This process must begin early, and should never be lost sight of during the student's entire mathematical experience.

There has been unquestionably a decline in the quality and scope of mathematical instruction in the elementary schools during the past quarter century. It is not our purpose to lay the blame for this, if such it is, upon any one in particular. The question is lost in the more general problem of overcrowded curricula. There is little doubt that the arithmetic of our fathers has been elbowed out by the study of history and English. But this may have been necessary. Under existing conditions it is of doubtful value to urge that a very large share of the child's time and attention be given to mathematics in the elementary school. And just here we encounter a difficulty which makes it impossible to draw final conclusions at this time.

Other elementary-school studies, particularly English, are concerned chiefly with the student's immediate environment. Mathematics involves training in a field which is largely foreign to his daily experience. To just this extent it is *forced* upon him. We should not, of course, lose sight of the appeal which manipulation of number symbols and other symbols makes to the imagination. But this is admittedly small in comparison with the demands of physical necessity, or the desire for self-expression. We teachers of mathematics are looking to the future of our pupils. Our science and art are primarily instrumentalities for which they do not instinctively recognize a need. This places our subjects in the category called "disciplinary," assuming this term to signify "directed toward an end not immediately present

to the consciousness of the pupil." Twenty-five years ago all schooling was largely disciplinary in this sense. At present much of it is anything but that. Ideals and modes of discipline are out of fashion (or were until eight months or so ago). It is unnecessary to justify this statement. One need only to state that while the aim of modern educational method is self-expressive, the ideal of discipline is self-repressive.

The natural result of this change of ideal was, as we all recall, a perfect orgy of inaccuracy and carelessness. Self-expression reached the point where much was expressed that might better have been left concealed. The results are seen to-day, not only in our own particular field, but in the daily press and in certain prevailing modes of fashion. "Educators" who had previously bombarded the teaching forces with "methods of development" and "inductive plans" were not slow to sense this danger. But their sole remedy seems thus far to have been found in incessant drill in the fundamental operations of numbers. This has reached the point where tests in computation are standardized, and progress is determined by these standards. If the whole of the arithmetic of the earlier generation were covered in this fashion, the process would require fifteen or twenty years, at least. So naturally the next stage in this interesting evolution is reached by a most extended list of omissions. In a recently issued state report, it was recommended that, except for decimals, the only fractions taught in the elementary school should be halves, fourths, eighths and sixteenths, with thirty-seconds for the brighter pupils. In many schools square root, proportion, greatest common divisor, lowest common multiple, numerical factors, circular measure and the fundamental principles of multiplication, division and fractions are not taught at all! Small wonder that in high school we find it difficult to extend these topics into the field of negative number and literal expression.

The remedy for existing conditions is not easy to find. It cannot be reduced to a few well-chosen sentences. But it probably lies along certain lines of action which at least are not mutually inconsistent.

(*a*) Accuracy should be sought not by endless repetition, but by a return in some degree to disciplinary methods in beginning arithmetic.

(*b*) Immediate interest as affording the teacher a mode of attack must to some extent give way to emphasis upon the importance of general principles.

(*c*) Percentage and its applications should be dethroned as the be-all and end-all of the higher elementary grades. The fact of its widespread use in commercial transactions should not overshadow its comparatively minor mathematical significance, but should be considered as one aspect of the problem of vocational training. Incidentally it may be noted that such training, to be a success, can hardly be expected to end with the eighth school year. If anywhere short of a complete high-school course, it can hardly be completed within a total of ten years. This leaves time for a thorough course in *Commercial Arithmetic* in the ninth year for those whose vocational choice is commerce, rather than, say, housekeeping, or aviation.

(*d*) Between the elementary arithmetic and the orthodox course in secondary algebra there should be given a distinct course in mathematics. The question of the make-up of this course might well be referred to a committee with instructions to report in six years. Let us devote a few minutes to the consideration of the present status of this part of the main problem.

We may as well understand at the outset that certain geometrical facts and certain algebraic relations of quantity are of as much importance mathematically, psychologically and practically, as is trade discount to the third remove, or the ultimate partiality of partial payments. Of these, sequence, direction, symmetry, congruence, similarity, area, average and transformation in general may be specified. There is also the important concept of definite magnitude independent of its numerical measure—of such a character that the measure depends equally upon the magnitude and upon the unit. These concepts are for the most part left without explicit enunciation until the conclusion of the high-school course, or omitted entirely. They seem to belong in the elementary course. Here they should be based neither upon purely empirical nor purely rational processes, but upon a wise combination of the two, in glorious disregard of Euclidean type forms.

The question of negative number demands our attention. It may well be that our problem would be greatly simplified by a

postponement of this topic to the period of formal algebra. Apparently much of our difficulty lies here. We have from the outset employed the *mechanism* of negative number in our algebraic processes to the exclusion of its *significance*. In so doing we have given the algebra an artificial character which it ought not possess. This concept is difficult enough in its simplest forms. In the form in which it appears, it passes entirely beyond the powers of realization of the student. It may well be that it stands in the way of his appreciation of the meaning of other equally important mathematical principles. The Greeks, to say the least, got on very well without it.

In such a course, the *principles* of arithmetic may well be brought into clear relief. At the same time the value of literal and other algebraic symbolism may be set forth with vigor and effect. Emphasis would naturally be placed upon the derivation and evaluation of formulæ, and ultimately upon various transformations of formulæ.

The problem of relative value of respective fields of application demands exhaustive study. In the past, as has been said, methods of commerce and exchange have displaced other considerations in the elementary curriculum. All admit the need of a change in this regard—but to what? The very difficult question of differentiating groups of pupils along vocational lines is encountered here. So many administrative problems are involved that it seems impracticable to offer definite recommendations at this time. The synthesis of applied problems must depend upon group analysis of detailed demands of the various vocations. It is possible, however, that a mathematical study of these demands may be of value in the work of vocational guidance.

At the time of the appointment of this committee, reference was made to the curriculum of the junior high school. The organization of these schools is primarily an administrative question. It would be most unfortunate if the reform of elementary-school mathematics were postponed until these schools shall be organized throughout the country. There is no essential reason for discriminating between the work of a teacher with a class in a junior high school, and the work of the same teacher teaching the same class at the same school age in an elementary school;

nor is there any apparent reason why the work of the seventh and eighth elementary grades should not, under existing conditions, be to some extent departmentalized. So much has been said in favor of the organization of the seventh, eighth and ninth years into an administrative unit that endorsement of the junior high-school plan would be superfluous at this time. It is sufficient that emphasis be placed upon the maintenance of continuity throughout the mathematics of the entire school course.

In conclusion, we would respectfully submit to our friends who are psychologically minded the desirability of a study of progressive standards of intricacy of abstract process, and of the power of generalization, as well as of accuracy.

William James has set the aim of philosophy as the ability to know a good man when one sees him. It may well be *one* aim of a properly ordered mathematical course to know a mathematical principle when one sees it. Is it too much to hope that a sound mathematical training may aid our boys and girls in recognizing a *moral* principle when they see it?

GEORGE B. GERMANN,
HOWARD F. HART,
AGNES LONG,
JOHN H. MINNICK,
C. P. SCOBORIA,
HARRISON E. WEBB, *Chairman.*

A GEOMETRIC REPRESENTATION.*

By E. D. Roe, Jr.

§ 1. Introduction.

In order to visualize the complex values of y, when such exist, of a plane curve $y=f(x)$, or a surface $y=f(x, z)$, and also for the purpose of representing some real curves in space by a single independent equation in x and y, I adjoin an ordinary complex plane, perpendicular to the x axis of the real xy plane, with its real axis parallel to the y axis, in fact always in the real plane and with its origin in the axis of x, so that the complex plane slides along always perpendicular to the x axis, OX, and at distance x from O the origin of the xy plane, as x changes. By this representation the equation of every curve or surface has an actual and uninterrupted locus from $-\infty$ to $+\infty$, including the usual real locus of $y=f(x)$ or $y=f(x, z)$, and some real curves in space can be represented by a single independent equation between two variables x and y. I prefer this method to that of projection used by Phillips and Beebe and others, because that method shows both the real curve and the complex portion in the same (real) plane and intersecting in points in which they do not intersect. Moreover, projection on a plane would be of no use in the case of surfaces. Projection on a fixed plane perpendicular to the x axis in case the complex curves lie on a cylinder whose axis is parallel to OX would not give a one-to-one correspondence and hence is not always available. The reason why we do not ordinarily get the complete locus, that is, the usual real and complex portion taken together, is that we have arbitrarily shut out the complex portion by the (usual real) kind of representation that we have chosen to use. While forms of representation for the complex portions of curves are known, it is believed that the method here used of representing surfaces and of visualizing the portions of their complex loci is new, as well as the representation of certain real

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curves in space by a single independent equation between x and y.

It is easily seen that the complete locus of the equation $x^2 + y^2 = a^2$ is, in addition to the circle, an equilateral hyperbola $x^2 - y^2 = a^2$ touching the circle at $(\pm a, 0)$ but turned out of the real plane through an angle of 90° about OX. The imaginary asymptotes $x^2 + y^2 = 0$, of the circle, are the actual asymptotes of the equilateral hyperbola and lie in its plane and pass through the real point (0, 0), of the real plane. Similarly $x^2 - y^2 = a^2$ has for its complex locus the circle $x^2 + y^2 = a^2$ turned through 90°. In fact $x^2 + y^2 = a^2$ and $x^2 - y^2 = a^2$ are complementary to each other when one is turned through an angle of 90°. In a similar way

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

are complementary. Complementary to $y^2 = 2mx$ is a congruent parabola turned through 90° about OX and 180° about OY, and touching the former at (0, 0). The cissoid

$$y^2 = \frac{x^3}{2a - x} \quad \text{and} \quad y^2 = \frac{x^3}{x - 2a}$$

are similarly complementary in the same way, cusp touching cusp point on at a common tangent. Likewise the lemniscate $(x^2+y^2)^2 = a^2(x^2-y^2)$ and the curve $(x^2-y^2)^2 = a^2(x^2+y^2)$ are complementaries when one is turned through 90° about OX. They touch at $(\pm a, 0)$.

The complex values of y as function of x have in general a different geometric representation from the complex values of x as function of y. This means only that the impossibility of being in the real plane exists in different ways along the two axes. In the case of some functions there is only one representation, no impossibility existing in the other direction. In the case of others the representation is of the same kind in both directions; y is the same function of x that x is of y. We consider the representation of y as function of x in this paper. If we consider complex representations in both directions "complete locus" will be accordingly enlarged. In the above illustrations we considered only the complete locus of y as function

of x. For surfaces analogous statements hold. In this paper it is shown how to obtain the equation of any surface in a form adapted to this representation, the equation of a family of spirals on a surface, the equation of the surface on which a family of spirals lies, and the length of an arc on a spiral, with illustrations. Next the application of the representation to the investigation of the function $y=(x^{x-1}(x-1)^x)^{1/(1-2x)}$, brought to my attention by Professor F. W. Very, and for which a graph was required, forms the major part of the paper. The special properties, the value systems with their sequence and contiguity, and the spiral systems of the function are considered in detail.

§ 2. The Equation of a Surface Adapted to this Representation.

In rectangular co-ordinates the equation of the surface is $f(x, y, z)=0$. y and z are the real and purely imaginary parts of the complex y represented in § 1. The equation of the surface in this representation is $y=|y|e^{\phi i}$.

$$|y|=(y^2+z^2)^{\frac{1}{2}}=((F_1(x, \phi))^2+(F_2(x, \phi))^2)^{\frac{1}{2}}$$

where

$$y=F_1(x, \phi), \quad z=F_2(x, \phi),$$

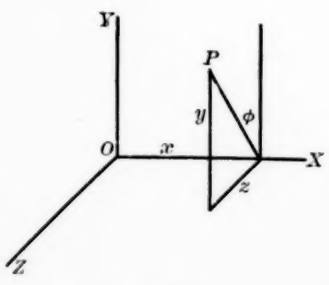

Fig. 1.

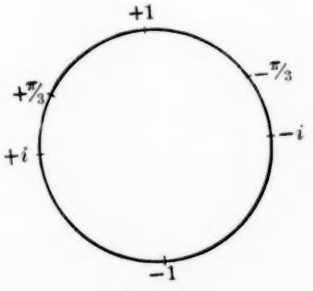

Fig. 2.

ϕ is the angle from the plane XOY to the plane XOP, Fig. 1; it is the amplitude or argument of the complex y of our representation. ϕ and x are independent variables. $z=y\tan\phi$, and $y=z\operatorname{ctn}\phi$. $F_1(x, \phi)$ is found by solving $f(x, y, y\tan\phi)=0$ for y, $F_2(x, \phi)$ by solving $f(x, z\operatorname{ctn}\phi, z)=0$ for z. The case where these solu-

tions are algebraically or practically impossible is left for consideration in a future paper. The equation of the surface is then

$$y=((F_1(x,\phi))^2+(F_2(x,\phi))^2)^{\frac{1}{2}}e^{\phi i} \qquad (1)$$

and it is seen that the locus exists from $x=-\infty$ to $x=\infty$.

Examples.

1. The equation of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

is

$$y=bc\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}\left(\frac{1}{c^2+b^2\tan^2\phi}+\frac{1}{c^2\operatorname{ctn}^2\phi+b^2}\right)^{\frac{1}{2}}e^{\phi i}, \qquad (2)$$
$$=R(x,\phi)e^{\phi i}.$$

2. The equation of the bi-parted hyperboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

is

$$y=bc\left(\frac{x^2}{a^2}-1\right)^{\frac{1}{2}}\left(\frac{1}{c^2+b^2\tan^2\phi}+\frac{1}{c^2\operatorname{ctn}^2\phi+b^2}\right)^{\frac{1}{2}}e^{\phi i}, \qquad (3)$$
$$=S(x,\phi)e^{\phi i}.$$

3. From examples 1 and 2 we note $S=iR$, $R=-iS$. From these examples follow easily the representation of the complete ellipsoid and the complete bi-parted hyperboloid. For the ellipsoid $y=R(x,\phi)e^{\phi i}$ when $|x|>|a|$ the equation becomes $y=iS(x,\phi)e^{\phi i}=S(x,\phi)e^{(\phi+\frac{\pi}{2})i}$, and for the bi-parted hyperboloid $y=S(x,\phi)e^{\phi i}$—when $|x|<|a|$, the equation takes the form

$$y=iR(x,\phi)e^{\phi i}=R(x,\phi)e^{(\phi+\frac{\pi}{2})i}.$$

Since $y=S(x,\phi)e^{(\phi+\frac{\pi}{2})i}$ is the bi-parted hyperboloid (where each y has an amplitude $\phi+\frac{\pi}{2}$ instead of ϕ) turned through 90° about OX, and in the same manner $y=R(x,\phi)e^{(\phi+\pi)i}$ is the ellipsoid turned through 90°, it is shown that the ellipsoid has

for its complex portion a bi-parted hyperboloid in a space turned through 90° about OX, and that the bi-parted hyperboloid has an ellipsoid turned through 90° for its complex part. It is not to be supposed that the interpretations for the complex portions of all surfaces will be as simple as these or that all the ordinates together will be turned through 90°.

§ 3. Spirals on the Surface $f(x, y, z)=0$.

We have seen in § 2 that the equation of the surface $f(x, y, z)=0$ is in our representation

$$y=((F_1(x, \phi))^2+(F_2(x, \phi))^2)^{\frac{1}{2}}e^{\phi i}.$$

If now in the place of ϕ we substitute a function of x, $f(x)$, we shall evidently have the equation of a line on the surface expressed by the single independent equation in x and y,

$$y=((F_1(x, f(x)))^2+(F_2(x, f(x)))^2)^{\frac{1}{2}}e^{f(x)i}, \qquad (4)$$

for as x increases, $f(x)$ either (1) increases, (2) decreases, or (3) is constant.

(1) If x increases, the amplitude $f(x)$ increases, the point $P=(x, y)$ continually winds around the surface tracing a right-handed spiral when viewed from O looking towards OX.

(2) If $f(x)$ decreases, the spiral is left-handed.

(3) If $f(x)$ is constant, the spiral degenerates into the intersection of the surface by a plane embracing OX.

If $f(x)$ increases for a while and then decreases, the spiral is right-handed for a while and then reverses and goes on as a left-handed spiral. If $f(x)=\sin kx$, the spiral can be made to be a fluctuating wabbly curve on the surface varying in shape according to the value taken for k.

Examples.

1. A spiral on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

has the equation

$$y=R(x, f(x))e^{f(x)i}, \qquad (5)$$

and when x reaches the values $\pm a$, the spiral does not stop but goes on to the complementary bi-parted hyperboloid and goes to infinity on its surface.

2. Similarly the spiral

$$y = S(x, f(x))e^{f(x)i} \tag{6}$$

on the bi-parted hyperboloid goes on the surface of the complementary ellipsoid between $x = -a$ and $x = a$.

3. If $a = \infty$ in example 1, § 2,

$$y = bc\left(\frac{1}{c^2 + b^2\tan^2\phi} + \frac{1}{c^2\operatorname{ctn}^2\phi + b^2}\right)^{\frac{1}{2}} e^{\phi i}, \tag{7}$$

the equation of an elliptic cylinder whose axis is OX, and whose semi-axes are b and c. If $b = c = a$,

$$y = ae^{\phi i}, \tag{8}$$

the equation of a circular cylinder, as could have been written down directly without this derivation. The equation of a spiral on the elliptic cylinder whose axis is OX and whose semi-axes are a and b is

$$y = ab\left(\frac{1}{a^2 + b^2\tan^2 f(x)} + \frac{1}{a^2\operatorname{ctn}^2 f(x) + b^2}\right)^{\frac{1}{2}} e^{f(x)i} \tag{9}$$

and the equation of a spiral on a circular cylinder is

$$y = ae^{f(x)i} \tag{10}$$

and as a special case of the last the equation of the helix is

$$y = ae^{kxi}. \tag{11}$$

(To be continued.)

NEW BOOKS.

Infinitesimal Calculus. By F. S. CAREY. New York: Longmans, Green and Company. Section II. Pp. 221. Price $3.00.

The first section of this work, which was reviewed in an earlier issue, closed with Chapter XI. and this section begins with Chapter XII. on exponential and hyperbolic functions, inverse circular and hyperbolic functions. Chapter XIII. treats of the motion of a particle along an axis and Chapter XIV. of definite integrals and some general theorems. The six remaining chapters deal with polar co-ordinates, curvature, partial differentiation, double integration, expansion in power series, curve tracing, envelopes, evolutes, roulettes, differential equations, graphics and nomography.

The Geometrical Sections of Isaac Barrow. Translated with Notes by J. M. CHILD. Chicago: The Open Court Publishing Company. Pp. 218. Price $1.25.

The first surprise for the reader of this book will be in the opening paragraph which is "Isaac Barrow was the first inventor of the Infinitesimal Calculus; Newton got the main idea of it from Barrow by personal communication; and Leibnitz also was in some measure indebted to Barrow's work, obtaining confirmation of his own original ideas, and suggestions for their further development, from the copy of Barrow's book that he purchased in 1673." The second surprise will probably be when he finds how convincing the facts are.

The volume is number 3 of the *Open Court Classics of Science and Philosophy* and contains a portrait of the author. The translation is from a first edition copy and contains much added material in the introduction and notes.

Contributions to the Founding of the Theory of Transfinite Numbers. By GEORG CANTOR. Translated and provided with an Introduction and Notes by PHILIP E. B. JOURDAIN. Chicago: The Open Court Publishing Co. Pp. 211. Price $1.25 net.

This is the first number in the *Open Court Classics of Science and Philosophy* and besides Cantor's two classic memoirs contains a very illuminating introduction by Mr. Jourdain. It should prove of great value as an introduction to the study of the subject.

William Oughtred. A Great Seventeenth-Century Teacher of Mathematics. By FLORIAN CAJORI. Chicago: The Open Court Publishing Co. Pp. 100. Price $1.00 net.

Oughtred was not a mathematics teacher but, like many another Englishman, he was a minister of the gospel who was much interested

in the subject, and those who would know his influence on latter developments in the field of mathematics will do well to read this volume.

The Philosophy of Mathematics. By J. B. SHAW. Chicago: The Open Court Publishing Co. Pp. 206.

The author of this volume aims to reach those readers who have had just an ordinary college training in mathematics and the more difficult parts of the philosophy have therefore been omitted. Such students seldom have any adequate conception of the real fundamentals of mathematics either as to what it is or what it aims to do. Such a work as this volume should do much to improve this condition.

Principles of Secondary Education. By ALEXANDER INGLIS. Boston: Houghton, Mifflin Company. Pp. 741. Price $2.75 net.

This is without doubt one of the most important of recent books on education. The author has shown excellent judgment in bringing together the consensus of prevailing thought on educational topics, and it is both comprehensive and scholarly as well as in clear language.

Every teacher will gain much from reading it.

Junior High School Mathematics, Second Course. By WILLIAM LEDLEY VOSBURG and FREDERICK WILLIAM GENTLEMAN. New York: The Macmillan Company. Pp. x + 212. Price 90 cents.

This is an eighth-grade book, including arithmetic, geometric mensuration, and an introduction to algebra through the use of equations. It follows much the same idea as the First Course, which was reviewed in an earlier issue, emphasizing the useful side of arithmetic and avoiding needless complication and discarded topics.

Plane and Spherical Trigonometry. By LEONARD M. PASSANO. New York: The Macmillan Co. Pp. xv + 141.

The author has written a concise, yet easily used book. It is logically arranged, and, while giving plenty of material for practice, leaves the impression of being short.

Elements of Trigonometry, with Tables. By W. C. BRENKE. New York: The Century Co. Pp. vi + 160.

This book contains, in revised form, the chapters on plane and spherical trigonometry from the author's text-book on "Advanced Algebra and Trigonometry." It covers the usual topics, with a chapter on polar coördinates, complex numbers, hyperbolic functions, etc., and seems especially fitted to students in technical schools.

An interesting device is the giving of answers for all odd numbered exercises, allowing their use as desired.

Advanced Algebra. By W. C. BRENKE. New York: The Century Co. Pp. vii + 196.

This text contains, in revised form, the work on algebra from the author's book on "Advanced Algebra and Trigonometry." It reviews the more immediately useful parts of secondary algebra and adds the usual topics of the freshman course. Graphic methods are emphasized, and the book is so written that considerable latitude is left to the teacher. As a whole it seems very usable.

Scientific Method in the Reconstruction of Ninth Grade Mathematics. By H. O. RUGG and J. R. CLARK. Chicago: The University of Chicago Press. Pp. 189. Price $1.00 + postage.

This the first number of the second volume of the Supplementary Educational Monographs published in connection with *The School Review* and the *Elementary School Journal*. It should be in the hands of every teacher of secondary mathematics and of those responsible for the course of study, as it gives a thorough and valuable discussion of its subject.

The data on the most frequent types of errors and the use of timed practice tests to diagnose and correct such errors, and on the uses for the various topics in first-year algebra are especially timely.

Community Leaflets. Published by the Bureau of Education. Price five cents to one cent each, according to number ordered.

No. 13. Lesson A–16. Caste in India.
No. 14. Lesson B–17. The Development of a System of Laws.
No. 15. Lesson C–17. Custom as a Basis of Law.
No. 16. Lesson A–20. Private Control of Industry.
No. 17. Lesson B–21. National Standards and the Bureau of Standards.
No. 18. Lesson C–21. Before Coins Were Made.
No. 19. Lesson A–24. Concentration of Population in Great Cities.
No. 20. Lesson B–24. Building the Industrial City of Gary.
No. 21. Lesson C–25. A Seaport as a Center of Concentration of Population and Wealth.

Community Arithmetic. By BRENELLE HUNT. New York: American Book Company. Pp. viii + 277.

This book takes up a wide range of applications of arithmetic, showing the needs for different processes, and developing them as necessary. It might almost be called a text on industries considered from the mathematical side. The book is fitted for Junior High School or Upper Elementary Grades, when only arithmetic is wanted.

An Introduction to Science. By BERTHA M. CLARK. New York: The American Book Co. Pp. 494.

Laboratory Manual. By the same author. Pp. 203.

The text and manual together form the basis for a course in general science for high schools. The manual is in loose leaf form, and provides space for the pupils to answer questions concerning the experiment made.

The course is a broad one, covering one hundred experiments in physics, chemistry, and biology. These experiments are well chosen from the standpoint of arousing interest, of meeting life needs, and of being within the ability of the children.

The Cadet Manual. By Major E. Z. Steever III, U. S. A., and Major J. L. Frink, U. S. A. Official Handbook H. S. V. U. S. Philadelphia: J. B. Lippincott Co. Pp. xxxi + 317.

A most complete and usable book founded on the "Steever Plan," used so successfully in the high schools of Wyoming. It includes drill, use of the rifle, physical training, wall scaling, etc., besides full directions for installing the system in high schools.

Elementary Algebra. By Elmer A. Lyman and Albertus Darnell. New York: The American Book Co. Pp. vii + 503. Price $1.32.

This algebra approaches the subject through simple abstract work, with some problems brought into the earlier topics. It carries the subject through intermediate algebra and logarithms, omitting formidable complexities and giving ample practice, both oral and written, in the important processes. Problems are not stressed as much as in some of the other recent books.

Regents' Original Exercises in Plane Geometry. By Earle B. Elmore. Syracuse: C. W. Bardeen. 84 pp.

A compilation of original exercises in Plane Geometry given by the N. Y. State Board of Regents in the examinations of the past twenty years. The problems are classified under the last theorem necessary to complete the proof. This book should be a valuable aid to all teachers of geometry and especially useful in New York State. The low price of 25 cents places it within the reach of every pupil.

NOTES AND NEWS.

An announcement has just been received from Italy to the effect that the publishing house of D. Capozzi, of Palermo, is taking over the publication of Professor Gino Loria's Bollettino di Bibliographia e Storia delle Scienze Matematiche. This journal has been of great value to those interested in the bibliography of their subject.

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